Operations on fuzzy ideals of $\Gamma-$semirings

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Abstract

The purpose of this paper is to introduce different types of operations on fuzzy ideals of $\Gamma-$semirings and to prove subsequently that these oprations give rise to different structures such as complete lattice, modular lattice on some restricted class of fuzzy ideals of $\Gamma-$semirings. A characterization of a regular $\Gamma-$semiring has also been obtained in terms of fuzzy subsets.

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1 Introduction

If we remove the restriction of having additive inverse of each element in a ring then a new algebraic structure is obtained what we call a semiring. Semiring has found many applications in various fields. In this regard we may refer to Golan's [5] and Weinert's [6] monographs. Semiring arises very naturally as the nonnegative cone of a totally ordered ring. But the nonpositive cone of a totally ordered ring fails to be a semiring because the multiplication is no longer defined. One can provide an algebraic home, called $\Gamma-$semiring, to the nonpositive cone of a totally ordered ring. The notion of $\Gamma-$semiring was introduced by M.M.K.Rao [9] in 1995 as a generalization of semiring as well as of $\Gamma-$ring. Subsequently by introducing the notion of operator semirings of a $\Gamma-$semiring Dutta and Sardar enriched the theory of $\Gamma-$semirings. In

this connection we may refer to [3]. The motivation for this paper is the fact that $\Gamma-$semiring is a generalization of semiring as well as of $\Gamma-$ring and fuzzy concepts of Zadeh [10] has been successfully applied to $\Gamma-$rings and semirings by Jun et al [7] and Dutta et al [2], [1]. We define here some compositions of fuzzy ideals in a $\Gamma-$semiring and study the structures of the set of fuzzy ideals of a $\Gamma-$semiring. Among other results we have deduced that sets of fuzzy left ideals and fuzzy right ideals form a zero-sum free semiring with infinite element. We have also deduced that fuzzy ideals of a $\Gamma-$semiring is a complete lattice which is modular if every fuzzy ideal is a fuzzy k-ideal.

2 Preliminaries

Definition 2.1 [9]*Let S and Γ be two additive commutative semigroups. Then S is called a $\Gamma-$semiring if there exists a mapping*
$S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:
(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Further, if in a $\Gamma-$semiring, $(S, +)$ and $(\Gamma, +)$ are both monoids and
(i) $0_S\alpha x = 0_S = x\alpha 0_S$
(ii) $x0_\Gamma y = 0_S = y0_\Gamma x$ for all $x, y \in S$ and for all $\alpha \in \Gamma$ then we say that S is a $\Gamma-$semiring with zero.

Throughout this paper we consider $\Gamma-$semiring with zero. For simplification we write 0 instead of 0_S and 0_Γ which will be clear from the context.

Definition 2.2 [10] *Let S be a non empty set. A mapping $\mu : S \to [0, 1]$ is called a fuzzy subset of S.*

Definition 2.3 [4]*Let μ be a non empty fuzzy subset of a $\Gamma-$semiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [fuzzy right ideal] of S if*
(i) $\mu(x + y) \geq min[\mu(x), \mu(y)]$ and
(ii) $\mu(x\gamma y) \geq \mu(y)$ [resp. $\mu(x\gamma y) \geq \mu(x)$] for all $x, y \in S, \gamma \in \Gamma$.

A fuzzy ideal of a $\Gamma-$semiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Definition 2.4 [5]*Let S be a non empty set and '+' and '.' be two binary operations on S, called addition and multiplication respectively. Then $(S, +, .)$ is called a hemiring (resp. semiring) if*

(i) (S, +) is a commutative monoid with identity element 0;
(ii) (S, .) is a semigroup (resp. monoid with identity element 1);
(iii) $a.(b+c) = a.b + a.c$ *and* $(b+c).a = b.a + c.a$ *for all* $a, b, c \in S$.
(iv) $a.0 = 0.a = 0$ *for all* $a \in S$;
(v) $1 \neq 0$.

A hemiring S is said to be zero-sum free if $a + b = 0$ *implies that* $a = b = 0$ *for all* $a, b \in S$.

An element a of a hemiring S is infinite iff $a + s = a$ *for all* $s \in S$.

For more on preliminaries we may refer to the references and their references.

3 Operations on fuzzy ideals

Throughout this paper unless otherwise mentioned S denotes a Γ-semiring with unities[3] and $FLI(S)$, $FRI(S)$ and $FI(S)$ denote respectively the set of all fuzzy left ideals, the set of all fuzzy right ideals and the set of all fuzzy ideals of the Γ-semiring S. Also in this section we assume that $\mu(0) = 1$ for a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) μ of a $\Gamma-$semiring ($\Gamma-$hemiring) S.

Definition 3.1 *Let S be a* Γ*-semiring and* $\mu_1, \mu_2 \in FLI(S)$ $[FRI(S),\ FI(S)]$. *Then the sum* $\mu_1 \oplus \mu_2$, *product* $\mu_1 \Gamma \mu_2$ *and composition* $\mu_1 \circ \mu_2$ *of* μ_1 *and* μ_2 *are defined as follows:*

$$(\mu_1 \oplus \mu_2)(x) = \sup_{x=u+v} [\min[\mu_1(u), \mu_2(v)] : u, v \in S]$$

$$= 0 \textit{ if for any } u, v \in S, u + v \neq x.$$

$$(\mu_1 \Gamma \mu_2)(x) = \sup_{x=u\gamma v} [\min[\mu_1(u), \mu_2(v)] : u, v \in S; \gamma \in \Gamma]$$

$$= 0 \textit{ if for any } u, v \in S \textit{ and for any } \gamma \in \Gamma,\ u\gamma v \neq x.$$

$$(\mu_1 \circ \mu_2)(x) = \sup_{x = \sum_{i=1}^{n} u_i \gamma_i v_i} [\min_{1 \leq i \leq n} [\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma]$$

$$= 0 \textit{ otherwise.}$$

Note. Since S contains 0, in the above definition the case $x \neq u + v$ for any $u, v \in S$ does not arise. Similarly since S contains left and right unity, the case $x \neq \sum_i u_i \gamma_i v_i$ for any $u_i, v_i \in S, \gamma_i \in \Gamma$ does not arise. In case of product of μ_1 and μ_2 if S has strong left or right unity [i.e., there exists $e \in S, \delta \in \Gamma$ such that $e\delta a = a$ for all $a \in S$] then the case $x \neq u\gamma v$ for any $u, v \in S$ and

for any $\gamma \in \Gamma$ does not arise. i.e., in otherwords there are $u, v \in S$ and $\gamma \in \Gamma$ such that $x = u\gamma v$.

Proposition 3.2 *Let* $\mu_1, \mu_2 \in FLI(S)[FRI(S), FI(S)]$. *Then* $\mu_1 \oplus \mu_2 \in FLI(S)[$ *resp.* $FRI(S), FI(S)]$.

Proof.
$$(\mu_1 \oplus \mu_2)(0) = \sup_{0=u+v} [\min[\mu_1(u), \mu_2(v)] : u, v \in S] \geq \min[\mu_1(0), \mu_2(0)] = 1 \neq 0.$$
Thus $\mu_1 \oplus \mu_2$ is non empty and $(\mu_1 \oplus \mu_2)(0) = 1$.
Let $x, y \in S$ and $\gamma \in \Gamma$. Then
$$\begin{aligned}
(\mu_1 \oplus \mu_2)(x+y) &= \sup_{x+y=p+q} [\min[\mu_1(p), \mu_2(q)] : p, q \in S] \\
&\geq \sup_{\substack{x=u+v \\ y=s+t}} [\min[\mu_1(u+s), \mu_2(v+t)] \text{: } u, v, s, t \in S] \\
&\geq \sup_{\substack{x=u+v \\ y=s+t}} [\min[\min[\mu_1(u), \mu_1(s)], \min[\mu_2(v), \mu_2(t)]] \text{: } u, v, s, t \in S] \\
&= \sup_{\substack{x=u+v \\ y=s+t}} [\min[\min[\mu_1(u), \mu_2(v)], \min[\mu_1(s), \mu_2(t)]] \text{: } u, v, s, t \in S] \\
&= \min[\sup_{x=u+v} [\min[\mu_1(u), \mu_2(v)]], \sup_{y=s+t} [\min[\mu_1(s), \mu_2(t)]]] \\
&= \min[(\mu_1 \oplus \mu_2)(x), (\mu_1 \oplus \mu_2)(y)].
\end{aligned}$$
Again
$$\begin{aligned}
(\mu_1 \oplus \mu_2)(x\gamma y) &= \sup_{x\gamma y=p+q} [\min[\mu_1(p), \mu_2(q)]] \\
&\geq \sup_{y=u+v} [\min[\mu_1(x\gamma u), \mu_2(x\gamma v)]] \\
&\qquad [\text{ Since } x\gamma y = x\gamma(u+v) = x\gamma u + x\gamma v] \\
&\geq \sup_{y=u+v} [\min[\mu_1(u), \mu_2(v)]] = (\mu_1 \oplus \mu_2)(y).
\end{aligned}$$
Hence $\mu_1 \oplus \mu_2 \in FLI(S)$.

Proposition 3.3 *Let* $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. *Then*

(i) $\mu_1 \oplus \mu_2 = \mu_2 \oplus \mu_1$.
(ii) $(\mu_1 \oplus \mu_2) \oplus \mu_3 = \mu_1 \oplus (\mu_2 \oplus \mu_3)$.
(iii) $\theta \oplus \mu_1 = \mu_1 = \mu_1 \oplus \theta$ *where* θ *is a fuzzy ideal of S, defined by,*
$$\theta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$
(iv) $\mu_1 \oplus \mu_1 = \mu_1$.
(v) $\mu_1 \subseteq \mu_1 \oplus \mu_2$ *and*
(vi) $\mu_1 \subseteq \mu_2$ *implies that* $\mu_1 \oplus \mu_3 \subseteq \mu_2 \oplus \mu_3$.

Proof. (i) We leave it as it follows easily.

(ii) Let $x \in S$.

$$((\mu_1 \oplus \mu_2) \oplus \mu_3)(x) = \sup_{x=u+v}[\min[(\mu_1 \oplus \mu_2)(u), \mu_3(v)] : u, v \in S]$$
$$= \sup_{x=u+v}[\min[\sup_{u=p+q}[\min[\mu_1(p), \mu_2(q)] : p, q \in S]], \mu_3(v)]$$
$$= \sup_{x=u+v}\ \sup_{u=p+q}[\min[\min[\mu_1(p), \mu_2(q)], \mu_3(v)]]$$
$$= \sup_{x=p+q+v}[\min[\mu_1(p), \mu_2(q), \mu_3(v)]].$$

Similarly we can deduce that $(\mu_1 \oplus (\mu_2 \oplus \mu_3))(x) = \sup_{x=p+q+v}[\min[\mu_1(p), \mu_2(q), \mu_3(v)]].$

Therefore $(\mu_1 \oplus \mu_2) \oplus \mu_3 = \mu_1 \oplus (\mu_2 \oplus \mu_3)$.

(iii) For any $x \in S$,

$$(\theta \oplus \mu_1)(x) = \sup_{x=u+v}[\min[\theta(u), \mu_1(v)], \text{ for } u, v \in S]$$
$$= \min[\theta(0), \mu_1(x)] = \mu_1(x).$$

Thus $\theta \oplus \mu_1 = \mu_1$. From (i) $\mu_1 \oplus \theta = \theta \oplus \mu_1 = \mu_1$.

(iv) Let $x \in S$. Then

$$(\mu_1 \oplus \mu_1)(x) = \sup_{x=u+v}[\min[\mu_1(u), \mu_1(v)], \text{ for } u, v \in S]$$
$$\leq \sup_{x=u+v}\mu_1(u+v) = \mu_1(x)$$

So $\mu_1 \oplus \mu_1 \subseteq \mu_1$

Again $\mu_1(x) = \min[\mu_1(0), \mu_1(x)]$

$$\leq \sup_{x=u+v}[\min[\mu_1(u), \mu_1(v)], \text{ for } u, v \in S] = (\mu_1 \oplus \mu_1)(x).$$

Therefore $\mu_1 \subseteq \mu_1 \oplus \mu_1$. Consequently, $\mu_1 = \mu_1 \oplus \mu_1$.

(v) Let $x \in S$. Then

$$(\mu_1 \oplus \mu_2)(x) = \sup_{x=u+v}[\min[\mu_1(u), \mu_2(v)], \text{ for } u, v \in S]$$
$$\geq \min[\mu_1(x), \mu_2(0)] = \mu_1(x).$$

Thus $\mu_1 \subseteq \mu_1 \oplus \mu_2$.

(vi) Let $\mu_1 \subseteq \mu_2$. and $x \in S$. Then

$$(\mu_1 \oplus \mu_3)(x) = \sup_{x=u+v}[\min[\mu_1(u), \mu_3(v)], \text{ for } u, v \in S]$$
$$\leq \sup_{x=u+v}[\min[\mu_2(u), \mu_3(v)], \text{ for } u, v \in S] = (\mu_2 \oplus \mu_3)(x).$$

Hence $\mu_1 \oplus \mu_3 \subseteq \mu_2 \oplus \mu_3$.

Proposition 3.4 *Let* $\mu_1, \mu_2 \in FLI(S)[FRI(S), FI(S)]$. *Then* $\mu_1 \circ \mu_2 \in FLI(S)$ *[resp. FRI(S), FI(S)].*

Proof. Since $(\mu_1 \circ \mu_2)(0)$

$$= \sup_{0=\sum_{i=1}^{n} u_i\gamma_i v_i}[\min_{1\leq i\leq n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$
$$\geq \min[\mu_1(0), \mu_2(0)] = 1 \neq 0 \text{ [Since } \mu_1(0) = \mu_2(0) = 1],$$

it follows that $\mu_1 \circ \mu_2$ is nonempty and $(\mu_1 \circ \mu_2)(0) = 1$.
Now, for any $x, y \in S$,
$$(\mu_1 \circ \mu_2)(x+y)$$
$$= \sup_{x+y=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$
$$\ge \sup[\min_{\substack{1 \le i \le m \\ 1 \le k \le l}} [\min[\min[\mu_1(u_i), \mu_2(v_i)], \min[\mu_1(p_k), \mu_2(q_k)]]] :$$
$$x = \sum_{i=1}^{m} u_i\gamma_i v_i, y = \sum_{k=1}^{l} p_k\gamma_k q_k, u_i, v_i, p_k, q_k \in S; \gamma_i \in \Gamma; m, l \in Z^+]$$
$$= \min[\sup_{x=\sum_{i=1}^{m} u_i\gamma_i v_i} [\min_{1\le i\le m}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, m \in Z^+],$$
$$\sup_{y=\sum_{k=1}^{l} p_k\gamma_k v_k} [\min_{1\le k\le l}[\min[\mu_1(p_k), \mu_2(q_k)]] : p_k, q_k \in S, \gamma_k \in \Gamma, l \in Z^+]]$$
$$= \min[(\mu_1 \circ \mu_2)(x), (\mu_1 \circ \mu_2)(y)].$$
Now $(\mu_1 \circ \mu_2)(x\gamma y)$
$$= \sup_{x\gamma y=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$
$$\ge \sup_{y=\sum_{j=1}^{m} s_j\delta_j t_j} [\min_{1\le j\le m}[\min[\mu_1(x\gamma s_j), \mu_2(t_j)]]]$$
$$\ge \sup_{y=\sum_{j=1}^{m} s_j\delta_j t_j} [\min_{1\le j\le m}[\min[\mu_1(s_j), \mu_2(t_j)]]] = (\mu_1 \circ \mu_2)(y)$$
Hence $\mu_1 \circ \mu_2 \in FLI(S)$

Proposition 3.5 *Let $\mu_1, \mu_2 \in FLI(S)[FRI(S), FI(S)]$. Then $\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2$.*

Proof. If for any $u, v \in S$ and for any $\gamma \in \Gamma$, $u\gamma v \neq x$ then $\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2$.
Now for any $x \in S, (\mu_1 \circ \mu_2)(x) =$
$$\sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$

$$\geq \sup_{x=u\gamma v}[\min[\mu_1(u),\mu_2(v)]] = (\mu_1\Gamma\mu_2)(x).$$

Thus $\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2$.

Proposition 3.6 *Let* μ_1 *be a fuzzy right ideal and* μ_2 *be a fuzzy left ideal of* S*. Then* $\mu_1\Gamma\mu_2 \subseteq \mu_1 \cap \mu_2$*.*

Proof. Let μ_1 be a fuzzy right ideal and μ_2 be a fuzzy left ideal of S. For $x \in S$,

$$(\mu_1\Gamma\mu_2)(x) = \sup_{x=u\gamma v}[\min[\mu_1(u),\mu_2(v)] : u, v \in S] \leq \sup_{x=u\gamma v}[\min[\mu_1(u\gamma v),\mu_2(u\gamma v)]]$$
$$\leq \sup_{x=u\gamma v}(\mu_1 \cap \mu_2)(u\gamma v) = (\mu_1 \cap \mu_2)(x).$$

Thus $\mu_1\Gamma\mu_2 \subseteq \mu_1 \cap \mu_2$.

The following is a characterization of a regular $\Gamma-$semiring in terms of fuzzy subsets.

Theorem 3.7 *A* $\Gamma-$*semiring* S *is multiplicatively regular*[9] *if and only if* $\mu_1\Gamma\mu_2 = \mu_1 \cap \mu_2$ *for every fuzzy right ideal* μ_1 *and every fuzzy left ideal* μ_2 *of* S*.*

Proof. Let S be a multiplicatively regular $\Gamma-$semiring and μ_1 be a fuzzy right ideal and μ_2 be a fuzzy left ideal of S. Then by Proposition 3.6, $\mu_1\Gamma\mu_2 \subseteq \mu_1 \cap \mu_2$. Let $c \in S$. Since S is multiplicatively regular, there exists an element x in S and $\gamma_1, \gamma_2 \in \Gamma$ such that $c = c\gamma_1 x\gamma_2 c$.

$$\text{Now } (\mu_1\Gamma\mu_2)(c) = \sup_{c=a\gamma b}[min[\mu_1(a),\mu_2(b)] : a, b \in S; \gamma \in \Gamma]$$
$$\geq min[\mu_1(c\gamma_1 x),\mu_2(c)] \qquad [\text{Since } c = (c\gamma_1 x)\gamma_2 c]$$
$$\geq min[\mu_1(c),\mu_2(c)] = (\mu_1 \cap \mu_2)(c).$$

Therefore $(\mu_1 \cap \mu_2) \subseteq \mu_1\Gamma\mu_2$ and hence $\mu_1\Gamma\mu_2 = \mu_1 \cap \mu_2$.

Conversely, let S is a $\Gamma-$semiring and for every fuzzy right ideal μ_1 and every fuzzy left ideal μ_2 of S, $\mu_1\Gamma\mu_2 = \mu_1 \cap \mu_2$. Let L and R be a left ideal and a right ideal of S respectively and let $x \in L \cap R$.

So $\lambda_L(x) = 1 = \lambda_R(x)$. Thus $(\lambda_L \cap \lambda_R)(x) = 1$. Now since $\lambda_R\Gamma\lambda_L = \lambda_R \cap \lambda_L$, so $(\lambda_R\Gamma\lambda_L)(x) = 1$. Therefore $\sup_{x=y\gamma z}[min[\lambda_R(y),\lambda_L(z)] : y, z \in S; \gamma \in \Gamma] = 1$.

Thus there exists some $r, s \in S$ and $\gamma_1 \in \Gamma$ such that $\lambda_L(s) = 1 = \lambda_R(r)$ for $x = r\gamma_1 s$. Then $r \in R$ and $s \in L$ and so $x = r\gamma_1 s \in R\Gamma L$. Therefore $L \cap R \subseteq R\Gamma L$. Also $L \cap R \supseteq R\Gamma L$. Thus $R\Gamma L = R \cap L$. Consequently, S is multiplicatively regular.

Proposition 3.8 *Let* $\mu_1, \mu_2 \in FI(S)$*. Then*

$$\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2.$$

Proof. By Proposition 3.5, $\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2$. For any $x \in S$, if $(\mu_1 \circ \mu_2)(x) = 0$ then obviously $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. Now for any $x \in S$,

$$(\mu_1 \circ \mu_2)(x)$$
$$= \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$
$$\le \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i\gamma_i v_i), \mu_2(u_i\gamma_i v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$
$$\le \min[\mu_1(x), \mu_2(x)] = (\mu_1 \cap \mu_2)(x).$$

Therefore $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. Again $(\mu_1 \cap \mu_2)(x) = \min[\mu_1(x), \mu_2(x)] \le \mu_1(x)$. Thus $\mu_1 \cap \mu_2 \subseteq \mu_1$. Similarly it can be shown that $\mu_1 \cap \mu_2 \subseteq \mu_2$. Hence the proposition.

Proposition 3.9 *Let $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. Then $\mu_1\Gamma\mu_2 \subseteq \mu_3$ if and only if $\mu_1 \circ \mu_2 \subseteq \mu_3$.*

Proof. Since $\mu_1\Gamma\mu_2 \subseteq \mu_1 \circ \mu_2$ it follows that $\mu_1 \circ \mu_2 \subseteq \mu_3$ implies that $\mu_1\Gamma\mu_2 \subseteq \mu_3$. Assume that $\mu_1\Gamma\mu_2 \subseteq \mu_3$. Let $x \in S$ and

$$x = \sum_{i=1}^{n} u_i\gamma_i v_i, u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+.$$

Then
$$\begin{aligned}\mu_3(x) &= \mu_3(\sum_{i=1}^{n} u_i\gamma_i v_i)\\ &\ge \min[\mu_3(u_1\gamma_1 v_1), \mu_3(u_2\gamma_2 v_2), ..., \mu_3(u_n\gamma_n v_n)]\\ &\ge \min[(\mu_1\Gamma\mu_2)(u_1\gamma_1 v_1), (\mu_1\Gamma\mu_2)(u_2\gamma_2 v_2), ..., (\mu_1\Gamma\mu_2)(u_n\gamma_n v_n)]\\ &\ge \min[\min[\mu_1(u_1), \mu_2(v_1)], ..., \min[\mu_1(u_n), \mu_2(v_n)].\end{aligned}$$

$$\mu_3(x) \ge \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]]] = (\mu_1 \circ \mu_2)(x).$$

Thus $\mu_1 \circ \mu_2 \subseteq \mu_3$.

Proposition 3.10 *Let $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. Then*

(i) $(\mu_1 \circ \mu_2) \circ \mu_3 = \mu_1 \circ (\mu_2 \circ \mu_3)$.

(ii) $\mu_1 \subseteq \mu_2$ implies that $\mu_1 \circ \mu_3 \subseteq \mu_2 \circ \mu_3$.

(iii) $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$, if S is commutative $\Gamma-$semiring.

(iv) $\mathbf{1} \circ \mu_1 = \mu_1$ where $\mathbf{1} \in FLI(S)$ is defined by $\mathbf{1}(x) = 1$ for all $x \in S$ [resp. $\mu_1 \circ \mathbf{1} = \mu_1, \mathbf{1} \circ \mu_1 = \mu_1 \circ \mathbf{1} = \mu_1$].

Proof. Proof of (i) follows from the definition.

(ii) Let $\mu_1 \subseteq \mu_2$. Now $(\mu_1 \circ \mu_3)(x)$

$$= \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_3(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$

$$\le \sup[\min_{1\le i\le n}[\min[\mu_2(u_i), \mu_3(v_i)]]] = (\mu_2 \circ \mu_3)(x).$$

Thus $\mu_1 \circ \mu_3 \subseteq \mu_2 \circ \mu_3$.

(iii) $(\mu_1 \circ \mu_2)(x)$

$$= \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$

$$= \sup_{x=\sum_{i=1}^{n} v_i\gamma_i u_i} [\min_{1\le i\le n}[\min[\mu_2(v_i), \mu_1(u_i)]]] \text{ if S is commutative } \Gamma-\text{semiring}$$

$= (\mu_2 \circ \mu_1)(x)$.

Hence $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$.

(iv) As S is with left unity $\sum_i [e_i, \delta_i] \in L$ which is defined by $\sum_i e_i\delta_i x = x$(*cf. Definition* 5.1[3]) for every $x \in S$ we have,

$$(\mathbf{1} \circ \mu_1)(x) = \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mathbf{1}(u_i), \mu_1(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$

$$= \sup[\min_{1\le i\le n}[\min[1, \mu_1(v_i)]]] = \sup[\min_{1\le i\le n} \mu_1[(v_i)]] \le \sup[\min_{1\le i\le n}[\mu_1(u_i\gamma_i v_i)]]$$

$$\le \mu_1(\sum_{i=1}^{n} u_i\gamma_i v_i) = \mu_1(x).$$

Therefore $(\mathbf{1} \circ \mu_1) \subseteq \mu_1$. Again $(\mathbf{1} \circ \mu_1)(x)$

$$= \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mathbf{1}(u_i), \mu_1(v_i)]] : u_i, v_i \in S; \gamma_i \in \Gamma, n \in Z^+]$$

$$\ge \min_{1\le i\le n}[\min[\mathbf{1}(e_i), \mu_1(x)]] \text{ [Since } \sum_j e_i\delta_i x = x]$$

$$= \mu_1(x)$$

So $\mu_1 \subseteq \mathbf{1} \circ \mu_1$ and hence $\mathbf{1} \circ \mu_1 = \mu_1$.

The following result shows that '.' distributive over '$\oplus$' from both sides.

Proposition 3.11 *Let* $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. *Then*
(i) $\mu_1 \circ (\mu_2 \oplus \mu_3) = \mu_1 \circ \mu_2 \oplus \mu_1 \circ \mu_3$, *and*
(ii) $(\mu_2 \oplus \mu_3) \circ \mu_1 = \mu_2 \circ \mu_1 \oplus \mu_3 \circ \mu_1$.

Proof. Since $\mu_2 \subseteq \mu_2 \oplus \mu_3$ therefore $\mu_1 \circ \mu_2 \subseteq \mu_1 \circ (\mu_2 \oplus \mu_3)$.
Similarly $\mu_1 \circ \mu_3 \subseteq \mu_1 \circ (\mu_2 \oplus \mu_3)$.
Thus $(\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3) \subseteq (\mu_1 \circ (\mu_2 \oplus \mu_3)) \oplus (\mu_1 \circ (\mu_2 \oplus \mu_3))$
$= (\mu_1 \circ (\mu_2 \oplus \mu_3))$.
Now let $x \in S$ be arbitrary.Then
$[\mu_1 \circ (\mu_2 \oplus \mu_3)](x)$

$$= \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_{1\le i\le n}[\min[\mu_1(u_i), (\mu_2 \oplus \mu_3)(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma, n \in Z^+]$$

$$= \sup[\min_{1\le i\le n}[\min[\mu_1(u_i), \sup_{v_i=r_i+s_i}[\min[\mu_2(r_i), \mu_3(s_i)]]]]]$$

$$= \sup_{x=\sum_{i=1}^{n}(u_i\gamma_i r_i + u_i\gamma_i s_i)} [\min_{1\le i\le n}[\min[\mu_1(u_i), \mu_2(r_i), \mu_3(s_i)]]$$

$$\le \sup_{x=\sum_{j=1}^{n} p_j\delta_j q_j + \sum_{k=1}^{m} p'_K\delta'_k q'_k} [\min[\min[\min_{1\le j\le}[\mu_1(p_j), \mu_2(q_j)]], \min[\min_{1\le k\le m}[\mu_1(p'_k), \mu_3(q'_k)]]]]$$

$$= \sup[\min[(\mu_1 \circ \mu_2)(u), (\mu_1 \circ \mu_3)(v)] : u = \sum_{j=1}^{n} p_j\delta_j q_j \text{ and } v = \sum_{k=1}^{m} pk'\delta'_k q'_k]$$

$= ((\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3))(x)$.
Thus $\mu_1 \circ (\mu_2 \oplus \mu_3) \subseteq (\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3)$.
Hence we conclude that $\mu_1 \circ (\mu_2 \oplus \mu_3) = (\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3)$.
Proof of (ii) follows similarly.

Theorem 3.12 *Let S be a $\Gamma-$semiring. Then FLI(S) and FRI(S)both are zero-sum free hemiring having infinite element 1 under the operations of sum and composition of fuzzy left ideals and fuzzy right ideals respectively.*

Proof. It is easy to see that $\theta \in FLI(S)$. Now by using Propositions 3.2, 3.3, 3.4, 3.10, 3.11 for any $\mu_1, \mu_2, \mu_3 \in FLI(S)$, we easily obtain
(i) $\mu_1 \oplus \mu_2 \in FLI(S)$,
(ii)$\mu_1 \circ \mu_2 \in FLI(S)$,
(iii) $\mu_1 \oplus \mu_2 = \mu_2 \oplus \mu_1$,
(iv) $\theta \oplus \mu_1 = \mu_1$,
(v) $\mu_1 \oplus (\mu_2 \oplus \mu_3) = (\mu_1 \oplus \mu_2) \oplus \mu_3$,
(vi) $\mu_1 \circ (\mu_2 \circ \mu_3) = (\mu_1 \circ \mu_2) \circ \mu_3$,
(vii) $\mu_1 \circ (\mu_2 \oplus \mu_3) = (\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3)$,
(viii) $(\mu_2 \oplus \mu_3) \circ \mu_1 = (\mu_2 \circ \mu_1) \oplus (\mu_3 \circ \mu_1)$.
Cosequently, FLI(S) is a hemiring under the operations of sum and composition of fuzzy ideals of S.
Now by Proposition 3.3(v), $\mathbf{1} \subseteq \mathbf{1} \oplus \mu$ for $\mu \in FLI(S)$.

Also $(\mathbf{1} \oplus \mu)(x) = \sup\limits_{x=y+z}[\min[\mathbf{1}(y), \mu(z)] : y, z \in S] \leq 1 = \mathbf{1}(x)$ for all $x \in S$.
Therefore $\mathbf{1} \oplus \mu \subseteq \mathbf{1}$ and hence $\mathbf{1} \oplus \mu = \mathbf{1}$ for all $\mu \in FLI(S)$.
Thus $\mathbf{1}$ is an infinite element of FLI(S). Now let $\mu_1 \oplus \mu_2 = \theta$ for $\mu_1, \mu_2 \in FLI(S)$. Then $\mu_1 \subseteq \mu_1 \oplus \mu_2 = \theta \subseteq \mu_1$. Consequently, $\mu_1 = \theta$. Similarly it can be shown that $\mu_2 = \theta$. Hence the hemiring FLI(S) is zero-sum free.
In analogous manner we can proof the result for FRI(S).

Remark. If S is a commutative $\Gamma-$semiring then FLI(S) and FRI(S) are semirings.

Corollary 3.13 *FI(S) is a zero-sum free simple semiring under the operations of sum and composition of fuzzy ideals.*

Proof. By Proposition 3.10(iv) we have $\mathbf{1} \circ \mu = \mu \circ \mathbf{1} = \mu$ for all $\mu \in FI(S)$. Hence the result follows from the above theorem.

Lemma 3.14 *Intersection of a nonempty collection of fuzzy left ideals (resp. fuzzy right ideals, fuzzy ideals) is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of S.*

Proof. Let $\{\mu_i : i \in I\}$ be a nonempty family of fuzzy ideals of S. Let $x, y \in S$.
Then $(\bigcap\limits_{i\in I} \mu_i)(x + y) = \inf\limits_{i\in I}[\mu_i(x + y)] \geq \inf\limits_{i\in I}[\min[\mu_i(x), \mu_i(y)]]$

$$= \min[\inf_{i\in I}[\mu_i(x)], \inf_{i\in I}[\mu_i(y)]] = \min[(\bigcap_{i\in I} \mu_i)(x), (\bigcap_{i\in I} \mu_i)(y)].$$

Again $(\bigcap\limits_{i\in I} \mu_i)(x\gamma y) = \inf\limits_{i\in I}[\mu_i(x\gamma y)] \geq \inf\limits_{i\in I}[\mu_i(y)] = (\bigcap\limits_{i\in I} \mu_i)(y)$.
Thus $\bigcap\limits_{i\in I} \mu_i$ is a fuzzy left ideal of S.
Similarly we can prove the other statements.

Theorem 3.15 *Let* μ_1 *and* μ_2 *be two fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of a* $\Gamma-$*semiring S. Then* $\mu_1 \oplus \mu_2$ *is the unique minimal element of the family of all fuzzy left ideals (resp. fuzzy right ideals, fuzzy ideals) of S containing* μ_1 *and* μ_2 *and* $\mu_1 \cap \mu_2$ *is the unique maximal element of the family of all fuzzy left ideals (resp. fuzzy right ideals, fuzzy ideals) of S contained in* μ_1 *and* μ_2*.*

Proof. Let$\mu_1, \mu_2 \in FLI(S)$. Then $\mu_1, \mu_2 \subseteq \mu_1 \oplus \mu_2$ [cf. Proposition 3.3(v)].
Suppose $\mu_1 \subseteq \psi$ and $\mu_2 \subseteq \psi$ where $\psi \in FLI(S)$. Now for any $x \in S$,
$(\mu_1 \oplus \mu_2)(x) = \sup\limits_{x=y+z}[\min[\mu_1(y), \mu_2(z)] : y, z \in S] \leq \sup[\min[\psi(y), \psi(z)]]$

$\leq \sup \psi(y+z) = \psi(x)$
Thus $\mu_1 \oplus \mu_2 \subseteq \psi$. Again $\mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2$.
Let us suppose that $\phi \in FLI(S)$ be such that $\phi \subseteq \mu_1$ and $\phi \subseteq \mu_2$. Then for any $x \in S$,
$(\mu_1 \cap \mu_2)(x) = \min[\mu_1(x), \mu_2(x)] \geq \min[\phi(x), \phi(x)] = \phi(x)$.
Thus $\phi \subseteq \mu_1 \cap \mu_2$. Uniqueness of $\mu_1 \oplus \mu_2$ and $\mu_1 \cap \mu_2$ with the stated properties are obvious.
Proofs of other cases follow similarly.

Theorem 3.16 *FLI(S) [resp. FRI(S), FI(S)] is a complete lattice.*

Proof. We define a relation '$\leq$' on FLI(S) as follows: $\mu_1 \leq \mu_2$ if and only if $\mu_1(x) \leq \mu_2(x)$ for all $x \in S$. Then FLI(S) is a poset with respect to '$\leq$'.By Theorem 3.15, every pair of elements of FLI(S) has lub and glb in FLI(S). Thus FLI(S) is a lattice. Now $\mathbf{1} \in FLI(S)$ and $\mu \leq \mathbf{1}$ for all $\mu \in FLI(S)$. So $\mathbf{1}$ is the greatest element of FLI(S). Let $\{\mu_i : i \in I\}$ be a non empty family of fuzzy left ideals of S. Then by Lemma 3.14, it follows that $\bigcap\limits_{i \in I} \mu_i \in FLI(S)$. Also it is the glb of $\{\mu_i : i \in I\}$. Hence FLI(S) is a complete lattice.
Proofs of other cases follow similarly.

Proposition 3.17 *If S is a $\Gamma-$semiring then the lattice $(FLI(S), \oplus, \cap)$ $[(FRI(S), \oplus, \cap), (FI(S), \oplus, \cap)]$ is modular if each of its member is a fuzzy left k-ideal [resp. fuzzy right k-ideal, fuzzy k-ideal].*

Proof. Let us assume that every member of FLI(S) is a fuzzy left k-ideal and $\mu_1, \mu_2, \mu_3 \in FLI(S)$ such that $\mu_2 \cap \mu_1 = \mu_2 \cap \mu_3, \mu_2 \oplus \mu_1 = \mu_2 \oplus \mu_3$ and $\mu_1 \subseteq \mu_3$. Then for any $x \in S$,
$\mu_1(x) = (\mu_1 \oplus \mu_1)(x) = \sup\limits_{x=u+v} [\min[\mu_1(u), \mu_1(v)] : u, v \in S]$
$\geq \sup[\min[\mu_1(u), (\mu_2 \cap \mu_1)(v)]] = \sup[\min[\mu_1(u), (\mu_2 \cap \mu_3)(v)]]$
$= \sup[\min[\mu_1(u), \min[\mu_2(v), \mu_3(v)]]] = \sup[\min[\min[\mu_1(u), \mu_2(v)], \mu_3(v)]]$
$\geq \sup[\min[\min[\mu_1(u), \mu_2(v)], \min[\mu_3(u+v), \mu_3(u)]]]$ [Since μ_3 is a left k-ideal].
$\geq \sup[\min[\min[\mu_1(u), \mu_2(v)], \min[\mu_3(u+v), \mu_1(u)]]]$
$= \sup[\min[\min[\mu_1(u), \mu_2(v)], \mu_3(u+v)]]$
$= \min[\sup[\min[\mu_1(u), \mu_2(v)]], \sup[\mu_3(u+v)]] = \min[(\mu_1 \oplus \mu_2)(x), \mu_3(x)]$
$= \min[(\mu_1 \oplus \mu_3)(x), \mu_3(x)] = \mu_3(x)$ [Since $\mu_3 \subseteq \mu_1 \oplus \mu_3$].
Thus $\mu_3 \subseteq \mu_1$ and hence $\mu_1 = \mu_3$. Hence $(FLI(S), \oplus, \cap)$ is modular.

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